

HARNACK, HÖLDER, GAUSS AND WIDDER: SERRIN'S PARABOLIC LEGACY

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ABSTRACT: James Serrin's fundamental contributions to the theory of quasilinear elliptic equations are well-known and widely appreciated. He also made less well-known contributions to the theory of quasilinear parabolic equations which we discuss in this note. Jürgen Moser gave greatly simplified proofs of the De Giorgi-Nash regularity results for linear divergence structure elliptic and parabolic differential equations using an original iterative technique. Serrin extended Moser's techniques and applied them to the study of divergence structure quasilinear elliptic and, in collaboration with Aronson, to divergence structure quasilinear parabolic equations. Specifically, among other results, they proved a maximum principle, Hölder continuity of generalized solutions and derived a Harnack principle for a very broad class of quasilinear parabolic equations. In subsequent work, Aronson applied these results to study non-negative solutions to divergence structure linear equations without regularity assumptions on the coefficients. The results include a two-sided Gaussian estimate for the fundamental solution and a generalization of the Widder Representation Theorem.

James Serrin's fundamental contributions to the theory of quasilinear elliptic equations [14] are well-known and widely appreciated. He also made less well-known but important contributions to the theory of quasilinear parabolic equations which we will describe here.

In two remarkable, essentially simultaneous works, Ennio De Giorgi [6] proved in 1956-57 the Hölder continuity of weak solutions of the divergence structure elliptic equation

$$\{A_{ij}(x)u_{x_i}\}_{x_j} = 0$$

and John Nash [11] proved in 1957-58 the same result for weak solutions of the parabolic equation

$$u_t - \{A_{ij}(x, t)u_{x_i}\}_{x_j} = 0.$$

Here and throughout this note we employ the convention of summation over repeated indicies. In case everything is independent of t , Nash also derives the De Giorgi result for elliptic equations. It should be noted that aside from boundedness, measurability and uniform ellipticity or parabolicity there are no further assumptions on the coefficients. Their work was totally independent and their methods completely different. Subsequently Jürgen Moser [9], [10]

introduced an iterative techniques to prove Harnack inequalities for both the elliptic (1961) and parabolic equations (1964) which enabled him to give greatly simplified proofs of the De Giorgi-Nash continuity results. In 1964 James Serrin [14] extended Moser's techniques to obtain a Harnack inequality and Hölder continuity for a broad class of divergence structure quasilinear elliptic equations of the form

$$\operatorname{div} \mathcal{A}(x, u, u_x) + \mathcal{B}(x, u, u_x) = 0.$$

We will be concerned with the extensions of Serrin's work to parabolic equations and its ramifications.

Let Ω be a bounded domain in \mathbf{R}^n and consider the space-time cylinder $Q = \Omega \times (0, T)$ for some fixed $T > 0$. We treat the second order quasilinear equation

$$u_t = \operatorname{div} \mathcal{A}(x, t, u, u_x) + \mathcal{B}(x, t, u, u_x), \quad (1)$$

where $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ is a given vector function of (x, t, u, u_x) , \mathcal{B} is a given scalar function of the same variables, and $u_x = (\partial u / \partial u_{x_1}, \dots, \partial u / \partial u_{x_n})$ denotes the spatial gradient of the dependent variable $u = u(x, t)$. Also here $\operatorname{div} \mathcal{A}$ refers to the divergence of the vector $\mathcal{A}(x, t, u(x, t), u_x(x, t))$ with respect to the variables (x_1, \dots, x_n) . The structure of (1) is determined by the functions $\mathcal{A}(x, t, u, p)$ and $\mathcal{B}(x, t, u, p)$. We assume that they are defined and measurable for all $(x, t) \in Q$ and for all values of u and p . In addition, \mathcal{A} and \mathcal{B} will be required to satisfy certain inequalities whose description follows.

A function $w = w(x, t)$ which is defined and measurable on Q belongs to the Bochner space $L^{p,q}(Q)$ if the iterated integral

$$\|w\|_{p,q} = \left\{ \int_0^T \left(\int_{\Omega} |w|^p dx \right)^{q/p} dt \right\}^{1/q}$$

is finite. Here p, q may be any real numbers ≥ 1 ; and with the obvious use of L^∞ norms rather than integrals we can allow p and q to have the value ∞ .

We shall always assume that the functions \mathcal{A} and \mathcal{B} satisfy inequalities of the form

$$\begin{aligned} p \cdot \mathcal{A}(x, t, u, p) &\geq a |p|^2 - b^2 u^2 - f^2 \\ |\mathcal{B}(x, t, u, p)| &\leq c |p| + d |u| + g \\ |\mathcal{A}(x, t, u, p)| &\leq \bar{a} |p| + e |u| + h. \end{aligned} \quad (2)$$

Here a and \bar{a} are positive constants, while the coefficients b, c, \dots, h are non-negative functions of (x, t) each contained in some space $L^{p,q}(Q)$, where p and q are non-negative real numbers (possibly different from coefficient to coefficient) such that

$$p > 2 \text{ and } \frac{n}{2p} + \frac{1}{q} < \frac{1}{2} \text{ for } b, c, e, f, h \quad (3)$$

and

$$p > 1 \text{ and } \frac{n}{2p} + \frac{1}{q} < 1 \text{ for } d, g. \quad (4)$$

The norms of the functions b, c, \dots, h in their respective spaces will be denoted simply by $\|b\|, \|c\|, \dots, \|h\|$. It is clear that there exists a positive number θ such that

$$p \geq \frac{2}{1-\theta} \text{ and } \frac{n}{2p} + \frac{1}{q} \leq \frac{1-\theta}{2} \text{ for } b, c, e, f, h$$

and

$$p \geq \frac{1}{1-\theta} \text{ and } \frac{n}{2p} + \frac{1}{q} \leq 1-\theta \text{ for } d, g.$$

The qualitative structure of equation (1) is entirely determined by the coefficients a and \bar{a} , the value of the norms $\|b\|$ through $\|h\|$ in their respective spaces and the numbers θ and n . A constant will be said to depend on the structure of equation (1) if it is determined by these quantities (and is uniformly bounded whenever these quantities are.)

The linear equation

$$u_t = \{A_{ij}(x, t)u_{x_i} + A_j(x, t)u + F_j(x, t)\}_{x_j} + B_j(x, t)u_{x_j} + C(x, t)u + G(x, t) \quad (5)$$

satisfies the preceding hypotheses if (i) the A_{ij} are bounded and measurable on Q and there exists a constant $\nu > 0$ such that $A_{ij}(x, t)\xi_i\xi_j \geq \nu|\xi|^2$ almost everywhere in Q and for all $\xi \in \mathbf{R}^n$; (ii) the coefficients A_j, B_j and F_j each belong to some space $L^{p,q}(Q)$ with p, q satisfying (3); and (iii) the coefficients C and G each belong to some space $L^{p,q}(Q)$ with p, q satisfying (4).

Without further hypotheses on \mathcal{A} and \mathcal{B} it is not possible, in general, to speak of a classical solution of equation (1), and it is correspondingly necessary to introduce the notion of generalized solution. Let $u = u(x, t)$ be a function which is locally of class $L^{2,\infty}$ in Q and possesses a strong derivative u_x which is locally of class $L^{2,2}$ in Q . Then u will be called a *weak solution* of equation (1) in Q if

$$\iint_Q \{-u\varphi_t + \varphi_x \cdot \mathcal{A}(x, t, u, u_x)\} dxdt = \iint_Q \varphi \mathcal{B}(x, t, u, u_x) dxdt \quad (6)$$

for any continuously differentiable function $\varphi = \varphi(x, t)$ having compact support in Q . It would be simpler to consider a less general class of weak solutions, namely those for which both u_x and u_t are locally of class $L^{2,2}$ in Q . However, from the point of view of existence theory the latter class is not as natural as the former. Thus even in the linear case some smoothness of the coefficients must be assumed in order to prove the existence of solutions having strong time derivatives.

In [5] we prove a global maximum principle, a local boundedness theorem and a Harnack inequality for weak solutions of equation (1) under the hypotheses listed above. All of these results depend on certain integral inequalities which are derived from the weak form of the differential equation. Moreover, in each case the proof depends on the iterative techniques introduced by Moser [9], [10] and further developed by Serrin [14] and Aronson & Serrin [4], [5]. In [4] iterative arguments are employed to prove a maximum principle for equations

whose structure inequalities are somewhat more general than those specified in (2). In [5] we also prove the interior Hölder continuity of weak solutions and to study the growth properties of non-negative solutions near certain parts of the boundary of their domain of definition.

As in [14], the fundamental inequalities are derived by using powers of u as test functions in the weak form (6) of the differential equation (1). For parabolic equations this presents certain technical difficulties which stem from the lack of an *a priori* bound on u and the lack of assumptions on the time derivative of u . These difficulties are resolved in great detail in [4] and [5]. Let $\eta = \eta(x, t)$ be a piecewise smooth non-negative function which vanishes in a neighborhood of the parabolic boundary Γ of Q . Then for almost all values of $\tau \in (0, T)$

$$\begin{aligned} \frac{1}{\beta+1} \int_{\Omega} \eta^2 \{ \bar{u}^{\beta-1} - (b+1)\kappa^{\beta}\bar{u} + \beta\kappa^{\beta+1} \} \Big|_{t=\tau} dx + \frac{a\beta}{2} \iint \eta^2 \bar{u}^{\beta-1} |\bar{u}_x|^2 dxdt \leq \\ \iint \mathfrak{F} \bar{u}^{\beta+1} dxdt + \frac{2}{\beta+1} \iint \eta |\eta_t| \bar{u}^{\beta+1} dxdt \end{aligned} \quad (7)$$

where $\beta \geq 1$ and $\kappa > 0$ are fixed constants and $\bar{u} = \max(0, u) + \kappa$. The integrations extend over $\Omega \times (0, \tau)$. Note that $\kappa = 0$ if f, g, h are zero. Moreover

$$\mathfrak{F} = F\eta^2 + 2G\eta|\eta_x| + H|\eta_x|^2$$

with

$$F = \beta \left(b^2 + \frac{f^2}{\kappa^2} \right) + \left(d + \frac{g}{\kappa} \right) + \frac{c^2}{a}, G = e \frac{h}{\kappa} \text{ and } H = \frac{4a'^2}{a}.$$

This inequality and its variants form the basis of the iterative arguments which are employed in [4] and [5]. The set $\Gamma = \{\partial\Omega \times [0, T)\} \cup \{\Omega \times \{t=0\}\}$ is called the parabolic boundary of Q and we say that $u \leq M$ on Γ if for every $\varepsilon > 0$ there is a neighborhood of Γ in which $u \leq M + \varepsilon$.

MAXIMUM PRINCIPLE: Let u be a weak solution of (1) in Q such that $u \leq M$ on Γ . Then almost everywhere in Q

$$u(x, t) \leq M + Ck,$$

where

$$k = (\|b\| + \|d\|) |M| + (\|f\| + \|g\|)$$

and C depends only on $T, |\Omega|$ and the structure of (1).

Actually

$$C = C(a, \|b\|, \|c\|, \|d\|, \theta, n, T, |\Omega|)$$

and so, in particular, is independent of a', e, f, g and h . Also the structural inequality $|\mathcal{A}(x, t, u, p)| \leq a' |p| + e |u| + h$ does not enter into the proof. All that is really needed is that the term $\varphi_x \cdot \mathcal{A}$ be integrable for φ_x belonging to $L^{2,2}$ locally in Q . Finally note that the maximum principle remains true if the differential equation (1) is replaced by the differential inequality

$$u_t \leq \operatorname{div} \mathcal{A}(x, t, u, u_x) + \mathcal{B}(x, t, u, u_x).$$

Under the same hypothesis as in the maximum principle there is also a minimum principle. Specifically, if $u \geq M$ on Γ then

$$u(x, t) \geq M - Ck$$

in Q .

The proof of the maximum principle involves iteration procedures based on the fundamental inequality (7) and one of its variants. Let $r = (\beta + 1)/2$ and $v = \bar{u}^r$. Introduce the norm

$$|||v||| = \sup \|v\|_{p', q'}$$

where the supremum is taken over all exponent pairs (p', q') whose Hölder conjugates (p, q) satisfy

$$\frac{n}{2p} + \frac{1}{q} \leq 1 - \theta \text{ and } p \geq \frac{1}{1 - \theta}.$$

Using (7), it is shown that

$$|||v^\sigma|||^{2/\sigma} \leq Cr^2 |||v|||^2$$

where $\sigma = 1 + 2\theta/n$ and C depends only on the structure of (1). For $m = 0, 1, 2, \dots$, set

$$\varphi_m = |||v|||^{2/r}$$

and rewrite the preceding inequality as

$$\varphi_{m+1} \leq C^{1/\sigma^m} \sigma^{2m/\sigma^m} \varphi_m.$$

Iteration yields

$$\varphi_{m+1} \leq C^{s_1} \sigma^{2s_2} \varphi_0$$

where

$$s_1 = \sum_{j=0}^m \sigma^{-j} \text{ and } s_2 = \sum_{j=0}^m j \sigma^{-j}.$$

In the limit as $m \rightarrow \infty$ we obtain an estimate for φ_0 in terms of $\|\tilde{u}\|_{2,\infty}$, $\|\tilde{u}_x\|_{2,2}$ and κ . The final result is obtained by using further iteration arguments based on a variant of the fundamental equality (7) to estimate $\|\tilde{u}\|_{2,\infty}$ and $\|\tilde{u}_x\|_{2,2}$.

In the absence of information about the boundary behavior of a weak solution of (1) it is possible to use iterative arguments similar to those used to prove the maximum principle to bound u in a subcylinder in terms of the norm $\|u\|_{2,2}$ over the full cylinder. Let (x', t') be a fixed point in the cylinder Q . Let $R(\rho)$ denote the open cube in \mathbf{R}^n with edge length ρ centered at x' and define

$$Q(\rho) = R(\rho) \times (t' - \rho^2, t').$$

The symbol $\|\cdot\|_{p,q,\rho}$ will be used to denote the $L^{p,q}$ norm of a function over the cylinder $Q(\rho)$.

LOCAL BOUNDEDNESS: Let u be a weak solution of (1) in Q and suppose that $Q(3\rho) \subset Q$. Then almost everywhere in $Q(\rho)$ we have

$$|u(x, t)| \leq C(\rho^{-(n+2)/2} \|u\|_{2,2,3\rho} + \rho^\theta k)$$

where C is a constant depending only on ρ and the structure of (1) and

$$k = \|f\| + \|g\| + \|h\|.$$

In particular, weak solutions of (1) are locally essentially bounded.

The Harnack inequality for non-negative harmonic functions gives a bound for the maximum of a harmonic function over an interior subset of the domain for definition in terms of the minimum over the same subset. For parabolic equations there is a similar result except that now the subsets must be separated by a non-empty time interval as explained in Moser's paper [10]. Moser deals with the linear equation

$$u_t - \{A_{ij}(x, t)u_{x_i}\}_{x_j} = 0$$

while in [5] the Harnack inequality is proved following the outline of Moser's proof but applied to the nonlinear equation (1).

Let (x', t') be a fixed point in the basic set Q . Let $Q(\rho)$ be as defined above and define

$$Q^*(\rho) = R(\rho) \times (t' - 8\rho^2, t' - 7\rho^2),$$

that is, $Q^*(\rho)$ translated downward a distance $7\rho^2$.

HARNACK INEQUALITY: Let u be a non-negative weak solution of (1) in Q . Suppose that $Q(3\rho) \subset Q$. Then

$$\max_{Q^*(\rho)} u \leq C \min_{Q(\rho)} (u + \rho^\theta k)$$

where C is a constant depending only on ρ and the structure of (1) and $k = \|f\| + \|g\| + \|h\|$.

Here \min and \max stand for the essential minimum and essential maximum, both of which are finite thanks to the local boundedness theorem. Moreover, C is independent of $\|f\|$, $\|g\|$ and $\|h\|$.

An important consequence of the local boundedness theorem and the Harnack inequality is the uniform Hölder continuity of solutions of equation (1). We write $X = (x, t)$, $Y = (y, s)$, etc. to denote points in space-time and introduce a pseudo-distance according to the definition

$$|X|^2 = \begin{cases} \max(x_i^2, -t/4) & \text{for } t \leq 0 \\ \infty & \text{for } t > 0 \end{cases}.$$

Thus the set $|Y - X| < \rho$ for fixed X is the cylinder

$$|x_i - y_i| < \rho, t - 4\rho^2 < s \leq t.$$

HÖLDER CONTINUITY: Suppose that u is a weak solution of (1) in Q . Then u is (essentially) Hölder continuous in Q . Moreover, if $|u| \leq L$ and X', Y' are points of Q with $s \leq t$ then

$$|u(Y) - u(X)| \leq H(L + k) \left(\frac{|Y - X|}{R} \right)^\alpha$$

where H and α are positive constants depending only on the structure of (1), $k = \|f\| + \|g\| + \|h\|$ and R is the pseudo-distance from X to the boundary of Q or $R = 1$ if this is smaller.

The proof of the Hölder continuity of u proceeds by an iteration argument based on the Harnack inequality applied to the oscillation of u . Since every weak solution of (1) is locally continuous it is meaningful to consider the value of a solution at a point. In particular, we can derive a pointwise Harnack inequality which makes explicit the dependence of the Harnack constant on the domain.

The results so far concern solutions defined in the space-time cylinder Q . They are also valid for solutions defined in the strip $S = \mathbf{R}^n \times (0, T)$. There is a drawback here, however, namely the requirement that the norms $\|b\|$ through $\|h\|$ be finite over the entire strip. Thus the results as stated do not apply to a solution in S when, for example, any of the quantities b through h is constant. To remedy this defect we introduce the set \mathcal{S} of all cylinders of the form $Q(\sigma)$, with $\sigma = \min(1, \sqrt{T})$, contained in S . A function $w = w(x, t)$ defined on S is said to belong to the class $L^{p,q}(\mathcal{S})$ if

$$\sup \|w\|_{p,q} < \infty$$

where the norms are taken over cylinders in the family \mathcal{S} . All of the preceding results continue to hold for solutions of (1) in S , where the coefficients in the structural inequalities are contained in the respective classes $L^{p,q}(\mathcal{S})$ rather than $L^{p,q}(Q)$.

POINTWISE HARNACK INEQUALITY: Let u be a non-negative weak solution of (1) in S , where the coefficients b through h in the structure inequalities are contained in the appropriate classes $L^{p,q}(\mathcal{S})$. Then for all points (x, t) and (y, s) in S with $0 < s < t < T$ we have

$$u(y, s) + k \leq \{u(x, t) + k\} \exp C \left(\frac{|x - y|^2}{t - s} + \frac{t}{s} \right).$$

Here $k = \sup \|f\| + \sup \|g\| + \sup \|h\|$ and C depends only on the structure of (1) and on T .

The proof of this result is based on the Harnack inequality and uses the fact that we can talk about the pointwise values of u . Note that the pointwise Harnack inequality holds for the linear equation (5) even when the coefficients are constant.

The following result which gives a lower bound on the growth of a solution of (1) in S as $t \searrow 0$ is a generalization of a result due to Nash [11] for the

linear equation $u_t - \{A_{ij}(x, t)u_{x_i}\}_{x_j} = 0$. It is a direct consequence of the pointwise Harnack inequality and plays an essential role in the Gaussian estimates discussed below.

LIMIT BEHAVIOR: Let u be a non-negative weak solution of (1) in the strip S , where the coefficients in (2) are contained in the appropriate classes $L^{p,q}(S)$. Suppose that for some $\alpha > 0$ we have

$$\mathcal{M} = \inf_{0 < t < T} \int_{|x|^2 < \alpha t} u(x, t) dx > 0.$$

Then there exist positive constants C_1 and C_2 such that

$$u(x, t) + k \geq C_1 t^{-n/2} \exp\left(-C_2 |x|^2 / t\right)$$

in S . Here C_1 depends only on $\alpha, \mathcal{M}, n, T$ and the structure of (1), while C_2 depends only on T and the structure of (1).

In [1], [2] and [3] the results of [5] are applied to the study of non-negative solutions of the linear equation (5). Many of the results in [1], [2] and [3] concern the properties of the weak fundamental solution $\Gamma(x, t; \xi, \tau)$ of the homogeneous equation

$$u_t = \{A_{ij}(x, t)u_{x_i} + A_j(x, t)u\}_{x_j} + B_j(x, t)u_{x_j} + C(x, t)u. \quad (8)$$

It is shown in [2] that the weak fundamental solution has all the essential properties of the classical fundamental solution and indeed that they coincide whenever the classical solution exists. Among the key results proved in [1] and [2] are the following bounds for Γ .

GAUSSIAN BOUNDS: Let $\Gamma(x, t; \xi, \tau)$ be the weak fundamental solution of a uniformly parabolic equation (8) whose coefficients are measurable and contained in the appropriate classes $L^{p,q}(S)$. Then there exist positive constants α_1, α_2 and C depending only on T and the bounds for the coefficients such that

$$\mathcal{C}^{-1} g_1(x - \xi; t - \tau) \leq \Gamma(x, t; \xi, \tau) \leq \mathcal{C} g_2(x - \xi; t - \tau)$$

for all $(x, t, \xi, \tau) \in S \times S$ with $t > \tau$, where $g_i(x, t)$ is the fundamental solution of the heat conduction equation $\alpha_i \Delta u = u_t$ for $i = 1, 2$.

It is noteworthy that these bounds do not require any smoothness assumptions on the coefficients of (8), unlike similar bounds which had previously appeared in the literature. The proof of the upper bound given in [1] and [2] is independent of the results in [5]. It is based on the Kolmogorov identity

$$\Gamma(x, t; \xi, \tau) = \int_{\mathbf{R}^n} \Gamma(x, t; \varsigma, \eta) \Gamma(\varsigma, \eta; \xi, \tau) d\varsigma$$

and a technical estimate for the growth at the center of a ball of a solution which is initially supported in the exterior of that ball. The the proof of the lower bound is based on the Limit Behavior Theorem. For the homogenous equation (8), $k = 0$ and the crux of the proof is the estimation of \mathcal{M} , which is carried out using the Pointwise Harnack Inequality.

Fabes and Stroock [7] made a deep study of Nash's methods and showed for the equation $u_t - \{A_{ij}(x, t)u_{x_i}\}_{x_j} = 0$ that the Gaussian bounds could be derived directly using an extension of Nash's ideas without first proving continuity and a Harnack principle. Indeed they showed that Hölder continuity and a Harnack inequality could be derived directly from the Gaussian bounds. Thus these three results-Hölder continuity, Harnack inequality and Gaussian estimates-are connected and are in some sense equivalent. Norris and Stroock [12] extended the results of [7] to more general equations.

Before continuing it is worth noting that in case of the equation $u_t - \{A_{ij}(x, t)u_{x_i}\}_{x_j} = 0$, if the coefficients A_{ij} are independent of t and if $n \geq 3$ then

$$\int_0^\infty \Gamma(x, t; \xi, 0) dt = G(x, \xi),$$

where $G(x, \xi)$ is the fundamental solution of the elliptic equation

$$\{A_{ij}(x)u_{x_i}\}_{x_j} = 0$$

As is remarked in [1], in this case the constants in the Gaussian estimate can be chosen independent of T so we can integrate these estimates over \mathbf{R}^+ to obtain

$$K^{-1} |x - \xi|^{2-n} \leq G(x, \xi) \leq K |x - \xi|^{2-n}.$$

This result was previously derived directly from potential theoretic considerations by Littman, Stampacchia and Weinberger [8] and by H. Royden [13].

A generalization of the Widder Representation Theorem for the heat conduction equation [15] is proved in [3]. This result gives a complete characterization of a non-negative solution of (8).

WIDDER REPRESENTATION THEOREM: Suppose that equation (8) is uniformly parabolic with measurable coefficients in the appropriate classes $L^{p,q}(S)$. If u is a non-negative weak solution of (8) in S then there exist a unique non-negative Borel measure ρ on R^n such that

$$u(x, t) = \int_{\mathbf{R}^n} \Gamma(x, t; \xi, 0) \rho(d\xi), \quad (9)$$

where Γ is the weak fundamental solution of (8) and ρ satisfies

$$\int_{\mathbf{R}^n} e^{-\sigma|x|^2} \rho(dx) < \infty \quad (10)$$

for some $\sigma > 0$. Moreover, the measure ρ is the initial trace of u , that is,

$$\lim_{t \searrow 0} \int_{\mathbf{R}^n} u(x, t) \psi(x) dx = \int_{\mathbf{R}^n} \psi(x) \rho(dx)$$

for all $\psi \in C(R^n)$ such that $|\psi(x)| \leq K e^{-\delta|x|^2}$ for some constant K and $\delta > \sigma$. Conversely, if u is given by (9) with a non-negative Borel measure ρ satisfying (10) then u is a non-negative weak solution of (8) with initial trace ρ .

Note that if ρ has a density μ then

$$u(x, t) = \int_{\mathbf{R}^n} \Gamma(x, t; \xi, 0) \mu(\xi) d\xi.$$

The existence and uniqueness of the representing measure ρ is proved in [2] and its characterization (10) is a consequence of the Gaussian lower bound. The proof that a function given by (9) with a measure satisfying (10) is a non-negative weak solution depends on an approximation argument and the Gaussian upper bound.

Serrin's seminal work on regularity of solutions of quasilinear elliptic equations [14] led directly to the extensions to quasilinear parabolic equations in [4] and [5]. These papers, in turn, gave rise to the detailed study of non-negative solution of linear parabolic equations described in [1], [2] and [3].

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